MATH 201: MIDTERM EXAM

WEEK 7

1. Basic Skills (40 points)

Problem 1.1 (8 points)**.** Circle the matrices which are in reduced row echelon form

$$A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_5 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_8 = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Problem 1.2 (8 points)**.** Suppose that A is a 3 by 4 matrix. If $\text{rank}(A|\vec{b}) = 2$, how many solutions does the equation $A\vec{x} = \vec{b}$ have? What is the *dimension* of the set of solutions?

Problem 1.3 (8 points)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \\ 2x \end{bmatrix}.$$

Write a matrix A satisfying $A\vec{x} = T(\vec{x})$.

Problem 1.4 (8 points)**.** Suppose that A is a square matrix. Suppose that A^2 is invertible. Is A invertible? Justify your answer.

Problem 1.5 (8 points)**.** Consider the matrix A shown below. Write a *basis* for the kernel of A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 2 & 1 & 0 & 6 \\ 3 & 0 & 1 & 10 \\ 4 & 1 & 0 & 10 \end{bmatrix}.$$

2. Typical Problems (60 points)

Problem 2.1. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the *orthogonal projection* onto the xy-plane. That is,

$$P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection across the plane $y = 0$. That is,

$$R \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -y \\ z \end{bmatrix}$$

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(a) (4 points) Find the matrix representing $T_1 = P \circ R$.
(b) (4 points) Find the matrix representing $T_2 = R \circ P$.
(c) (4 points) Find the image and kernel of T_1 and T_2.

Problem 2.2 (12 points)**.** Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Find *all* two-by-two matrices B such that $AB = BA$.

Problem 2.3 (12 points)**.** Find *all* two-by-two matrics A such that

- The *image* of A is span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
- The *kernel* of A is span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Problem 2.4. Let $\mathscr{B} = \{\vec{v}_1, \vec{v}_2\}$ where

$$\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Let $\vec{x} = \begin{bmatrix} -8 \\ -4 \\ 2 \end{bmatrix}$.

(a) (6 points) Is $\vec{x} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$?
(b) (6 points) Write $[\vec{x}]_{\mathscr{B}}$.
(c) (**Bonus:** 2 points) Find a matrix A such that $A\vec{x} = [x]_B$ for *all* $\vec{x}$ in $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

Problem 2.5. Let

$$W = \left\{(x, y, z, w) \in \mathbb{R}^4 \mid x + 2y + 3z = 0, \quad z = w\right\}.$$

(a) (6 points) Show that W is a subspace of $\mathbb{R}^4$? (justify your answer).
(b) (6 points) Find a basis for W and determine its dimension.

3. Challenge Problem

Recall that the *cross-product* of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$ is given by

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Problem 3.1 (**Bonus:** 8 points)**.** Fix a unit vector $\vec{u} \in \mathbb{R}^3$. Define a linear map

$$S(\vec{x}) = \vec{u} \times (\vec{u} \times \vec{x}) + \vec{x}.$$

(a) Find a matrix A such that $S(\vec{x}) = A\vec{x}$.
(b) Describe the image and kernel of S.
(c) Compute S^2.
(d) Give a simple geoemtric description of S.