New Uzbekistan University

Statement of Ethics: I agree to complete this exam without unauthorized assistance.

FULL NAME: SIGNATURE:

ID NUMBER: GROUP E.g. FSE1:

EXAM ROOM: M A T H

Final Exam

COURSE NAME: Linear Algebra COURSE CODE: Math 201

EXAMINATION DURATION: 90 MINUTES EXAM VERSION A

ADDITIONAL MATERIALS A simple scientific calculator is allowed. No other materials are allowed.

Please do not open the examination paper until directed to do so.

READ INSTRUCTIONS FIRST! VIOLATION OF THE RULES CAN LEAD TO A LOSS OF POINTS.

- Unless otherwise stated, you must **justify all your answers**.
- Your work must be neat and legible. **Circle your final answer.**

FOR INSTRUCTOR USE ONLY (DO NOT WRITE ANYTHING):

Question:	1.1	1.2	1.3	1.4	1.5	2.1	2.2	2.3	3.1	3.2	3.3	4.1
Points:												0
Bonus Points:												
Total:												

1 True/False (20 points)

Instructions: Decide whether each of the statements below is **true** or **false**. Circle your choice. Then, in the space below, **justify your answer**. You will receive a maximum of **1 point** for correctly answering "true" or "false". You will receive the remaining **3 points** for correct justification.

Problem 1.1 (4 points).

Suppose that A is a square matrix. If the linear system $A^2\mathbf{x} = \mathbf{b}$ is consistent then so is $A\mathbf{x} = \mathbf{b}$.

[✓ **True**] **False**

Justification:

Solution. True. If $A^2\mathbf{x} = \mathbf{b}$, set $\mathbf{y} = A\mathbf{x}$. Then $A\mathbf{y} = A^2\mathbf{x} = \mathbf{b}$, so $\mathbf{y}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Thus $\mathbf{b} \in \operatorname{Im}(A^2) \subseteq \operatorname{Im}(A)$.

Problem 1.2 (4 points).

There exists an invertible 2×2 matrix A such that $A^{-1} = A$.

[✓ **True**] **False**

Justification:

Solution. True. Taking $A = -I_2$ gives $A^{-1} = -I_2 = A$.

Problem 1.3 (4 points).

Let $f(t) = 7 + 3t + t^2$, $g(t) = 8 + 8t + 4t^2$, and $h(t) = 2 + 2t + t^2$. The set $\{f, g, h\}$ is linearly independent in P_2.

True [✓ **False**]

Justification:

Solution. False. Note that $g(t) = 4h(t)$

Problem 1.4 (4 points).

There exists a subspace V of $\mathbb{R}^5$ suich that $\dim(V) = \dim(V^\perp)$.

True [✓ **False**]

Justification:

Solution. False. We know that $\dim(V) + \dim(V^\perp) = n = 5$. Furthermore, dimensions are always integers. So since 5 is odd, this is impossible.

Problem 1.5 (4 points).

If A and B are 3×3 matrices with the same trace and the same determinant, then they have the same eigenvalues.

True [✓ **False**]

Justification:

Solution. False. Trace and determinant give only the sum and product of eigenvalues. For instance,

$$A = \mathrm{diag}(0, 1, 6), \qquad B = \mathrm{diag}(0, 2, 5)$$

both have trace 7 and determinant 0, but their eigenvalues differ.

2 Basic Skills (20 points)

Instructions: Each question in this section is worth 5 points. **Show your work!!!**

Problem 2.1 (5 points). Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution. Row reduce:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

There are **two pivot rows**, so

$$\text{rank}(A) = 2.$$

Problem 2.2 (5 points). Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$$

Write a basis for $\text{Im}(A)$ and a basis for $\ker(A)$.

Solution. All columns are multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so

$$\text{Im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Solve $x_1 + x_2 + x_3 = 0$. Let $x_2 = t$, $x_3 = s$, then $x_1 = -t - s$. Thus

$$\ker(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Problem 2.3 (5 points). Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

That is, find an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Solution. First find the eigenvalues of A. Since A is upper triangular, the eigenvalues are the diagonal entries:

$$\lambda_1 = 1, \qquad \lambda_2 = 2.$$

$$E_1 = \ker(A - I) = \ker\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$E_2 = \ker(A - 2I) = \ker\left(\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

Thus an eigenbasis for A is

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

Thus $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Problem 2.4 (5 points). Let

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

be a basis for $\mathbb{R}^2$. Find the $\mathcal{B}$–coordinate vector of $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Solution. Solve

$$a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Then $(a + b, b) = (3, 5)$, so $b = 5$ and $a = 3 - 5 = -2$. Thus

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

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3 Typical Problems (60 points)

Instructions: Each question in this section is worth **20 points**. **Show your work!!!**

Problem 3.1. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy

$$A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\3\end{bmatrix}, \qquad A\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \qquad A\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\2\\-2\end{bmatrix}.$$

(a) (6 points). Compute the matrix of A in the basis $\mathcal{B} = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$.

(b) (6 points). Compute all eigenvalues of A.

(c) (8 points). Determine all the subspaces of $\mathbb{R}^3$ that are *invariant* with respect to A. That is, find all the subspaces $V \subset \mathbb{R}^3$ such that whenever $\mathbf{v} \in V$, $A\mathbf{v} \in V$.

Solution. Let $\mathbf{b}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$.

(a) The columns of $[A]_{\mathcal{B}}$ are $[A\mathbf{b}_i]_{\mathcal{B}}$. We have

$$A\mathbf{b}_1 = \begin{bmatrix}3\\3\\3\end{bmatrix} = 3\mathbf{b}_1 \qquad A\mathbf{b}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix} = \mathbf{b}_2 \qquad A\mathbf{b}_3 = \begin{bmatrix}0\\2\\-2\end{bmatrix} = 2\mathbf{b}_3$$

Thus

$$[A]_{\mathcal{B}} = \begin{bmatrix}3&0&0\\0&1&0\\0&0&2\end{bmatrix}.$$

(b) The eigenvalues are the diagonal entries in this basis:

$$\lambda = 3,\ 1,\ 2,$$

each with geometric multiplicity 1.

(c) The invariant subspaces are

$$\begin{aligned} &\{\mathbf{0}\} \\ &\operatorname{span}\{\mathbf{b}_i\} \\ &\operatorname{span}\{\mathbf{b}_i, \mathbf{b}_j \mid i \neq j\} \\ &\operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \mathbb{R}^3. \end{aligned}$$

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Problem 3.2. Let W be the set of *all* matrices A such that

- $\det A = 2$
- **and** $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$

(a) (8 points). Is W a linear subspace? If yes, find its dimension.

(b) (6 points). Suppose that $A \in W$. What are the possible values of $\operatorname{tr} A$?

(c) (6 points). For which values of the trace does A have two real eigenvalues?

Solution.

(a) Suppose that $a, b, c, d \in \mathbb{R}$ satisfy

$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}.$$

The condition $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$ gives

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a+b\\c+d\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix},$$

so

$$a + b = 4, \qquad c + d = 2.$$

Use the two linear equations above to solve in terms of c. From $c + d = 2$ we get

$$d = 2 - c.$$

From $a + b = 4$ we get

$$a = 4 - b.$$

Plugging these in to $\det(A) = 2$,

$$(4-b)(2-c) - bc = 2.$$

Expanding,

$$(4-b)(2-c) - bc = 8 - 4c - 2b + bc - bc = 8 - 4c - 2b,$$

so

$$8 - 4c - 2b = 2 \implies 2b + 4c = 6 \implies \boxed{b = 3 - 2c.}$$

Then

$$a = 4 - b = 4 - (3 - 2c) \Rightarrow \boxed{a = 1 + 2c} \text{ and } \boxed{d = 2 - c}.$$

In other words,

$$W = \left\{ \begin{bmatrix}1+2c & 3-2c\\c & 2-c\end{bmatrix} \mid c \in \mathbb{R} \right\}.$$

This set is clearly **not** closed under addition and is therefore not a linaer subspace.

(b) by the above calculation, the trace of A is

$$\operatorname{tr}(A) = a + d = (1 + 2c) + (2 - c) = 3 + c.$$

As c can be any real number, $\operatorname{tr}(A)$ can be *any* real number. So all real numbers occur as possible traces.

(c) For a 2×2 matrix, the eigenvalues λ satisfy

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det(A) = 0.$$

Here $\det(A) = 2$, so the characteristic polynomial is

$$\lambda^2 - (\operatorname{tr} A)\lambda + 2 = 0.$$

So A has two distinct real eigenvalues when

$$(\mathrm{tr} A)^2 - (4 \cdot 1 \cdot 2) > 0 \iff (\operatorname{tr} A)^2 > 8 \iff |\operatorname{tr} A| > \sqrt{8}.$$

Problem 3.3 (20 points)**.** Consider the discrete dynamical system defined by

$$\mathbf{x}_{n+1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{x}_n,$$

Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute $\mathbf{x}_{50}$.

Solution. Note that $A = I + H$ where $H = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Also,

$$H^2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} A^2 &= (I+H)^2 = I + 2H \\ A^3 &= (I+2H)(I+H) = I + 3H \\ &\vdots \qquad \vdots \\ A^n &= I + nH. \end{aligned}$$

So

$$\begin{aligned} \mathbf{x}_{50} &= (I + 50H)\,\mathbf{x}_0 \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 50 \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 101 \\ 1 \end{bmatrix}. \end{aligned}$$

4 Challenge Problem: 10 Bonus Points

Instructions: This is a *challenge problem* and should only be attempted if you are finished with the rest of the exam. It is worth 10 bonus points.

Problem 4.1.
Suppose that A is a 4×4 matrix with entries $1, -1$, or 0. What is the *maximum* possible value of its determinant? Give an example of a matrix whose determinant is equal to this maximum value.

Solution. Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ be the rows of A. The Gram–Schmidt process implies that

$$|\det A| \leqslant \prod_{i=1}^{4} \|\mathbf{r}_i\|.$$

By assumption, the maximum *length* of each row vector is 4. Thus

$$|\det A| \leqslant \|\mathbf{r}_1\|\|\mathbf{r}_2\|\|\mathbf{r}_3\|\|\mathbf{r}_4\| \leqslant 16.$$

To construct an example, we just need to find 4 orthogonal vectors, whose components are all 0, 1, or -1, each with length 2. For example,

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

One easily checks that $\mathbf{r}_i \cdot \mathbf{r}_j = \mathbf{0}$. Therefore

$$AA^\top = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix} = \left[\mathbf{r}_i \cdot \mathbf{r}_j\right]_{ij} = I.$$

Thus

$$|\det\left(AA^\top\right)| = |\det A|^2 = \det(4I) = 4^4$$
$$|\det A| = 4^2 = 16.$$

END OF EXAMINATION.