LINEAR ALGEBRA: DEFINITIONS AND FACTS

SPRING, 2026

Week 1

Definitions:

(1) A **matrix** is a rectangular array of numbers made up of **rows** which are horizontal lists and **columns** which are vertical lists. The individual numbers in the matrix are called **entries**. The notation a_{ij} is used to indicate the entry which is in the ith row and the jth column.
(2) The **size** of a matrix A is $n \times m$ where n is the number of rows in A and m is the number of columns. A matrix is called **square** if the number of rows is equal to the number of columns.
(3) A square matrix with entries a_{ij} is called **diagonal** $a_{ij} = 0$ unless $i = j$. A square matrix is called **upper (or lower) triangular** if all entires on or below (above) the diagonal entries are equal to zero. If all the entries of a matrix are zero, it is called a **zero matrix**.
(4) A matrix with a single row is called a **row vector**.
(5) A matrix with a single column is called a **column vector** or simply **vector**.
(6) The set of all column vectors with n components is denoted by $\mathbb{R}^n$. We will refer to $\mathbb{R}^n$ as a **vector space**.
(7) A **component** is an entry in a row or column vector.
(8) The $n \times n$ matrix whose diagonal entries are equal to 1 and whose off-diagonal entries are all zero is called the **n by n identity matrix** and is denoted by I_n.
(9) A matrix is in **reduced row-echelon form (rref)** if it satisfies the following conditions.
    (a) If a row has nonzero entries, then the first nonzero entry is a 1. Such a 1 is called a **leading 1** or **pivot**.
    (b) If a column contains a leading 1, then all other entries in the column are 0.
    (c) Each leading 1 is to the *right* of any leading 1s in the row above.
    (d) If there exists a row of all 0s, it appears as the last row of the matrix. (This condition is implied by the others).
(10) The following three operations on rows are called **elementary row operations**.
    (a) Divide a row by a nonzero scalar (number).
    (b) Subtract a multiple of a row from another row.
    (c) Swap two rows.
(11) A **linear equation** is an equation of the form

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where each a_i is a number, and where x_i, y are variables.
(12) A **system of linear equations** is a collection of 2 or more linear equations.
(13) A system of linear equations is called **consistent** if it has *at least one* solution. A system is called **inconsistent** if it has *no solutions.*
(14) Given a system of n equations in m variables, that is,

$$\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\
\vdots &= \;\vdots \;+\; \vdots \;+ \cdots +\; \vdots \\
y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m,
\end{aligned}$$

the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

is called the **coefficient matrix** for the system. The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} & | & y_1 \\ a_{21} & a_{22} & \dots & a_{2m} & | & y_2 \\ & \vdots & \vdots & \ddots & | & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & | & y_n \end{bmatrix}$$

is called the **augmented matrix** for the system and is often denoted by $[A|\vec{y}]$.

(15) Note that the coefficient matrix of a system of linear equations in m variables has m columns, one for each variable. Let A be the coefficient matrix of such a system. The variable x_i is called **a free variable** if the ith column in $\mathrm{rref}(A)$ *does not contain a leading 1.*

(16) **Gauss–Jordan elimination** is an algorithm that transforms a matrix into reduced row echelon form through a finite sequence of elementary row operations. The algorithm is as follows.
   (a) Form the augmented matrix of the system of linear equations.
   (b) Beginning with the leftmost column, locate a nonzero entry and interchange rows if necessary to place it in the highest available row; this entry is the pivot.
   (c) Scale the pivot row so that the pivot entry equals 1.
   (d) Use row replacement operations to make all other entries in the pivot column equal to 0, both above and below the pivot.
   (e) Move to the next column to the right and the next row down, and repeat Steps 2–4 until no further pivots can be found.
   (f) Continue until the matrix is in reduced row echelon form, where each pivot column contains a single 1 and zeros elsewhere.

Facts:

(1) Linear equations represent lines, planes or their higher-dimensional analogues.
(2) A solution to a system of linear equations (if it exists) represents a point in the intersection of the lines or planes described by each equation in the system.
(3) The reduced row echelon form of a matrix does *not* depend on the sequence of elementary row operations used.

Week 2

Definitions:

(1) The **rank** of a matrix A is the number of leading 1s in its reduced row-echelon form (which is uniquely determined by A).
(2) Suppose that $\vec{v}$ and $\vec{w}$ are two vectors with the same number of components. Suppose that the components of $\vec{v}$ are $v_1, \dots, v_n$ and the components of $\vec{w}$ are $w_1, \dots, w_n$. The **dot product** of $\vec{v}$ and $\vec{w}$ is defined to be the scalar

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Note: this operation is defined even when $\vec{v}$ and $\vec{w}$ are vectors of different types. That is either can be either a column of row vector.

(3) If A is an $n \times m$ matrix with row vectors $\vec{w}_1, \dots, \vec{w}_n$, and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the **product of the matrix A with the column vector $\vec{x}$** is defined to be

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1- \\ \vdots \\ -\vec{w}_n- \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}.$$

(4) A vector $\vec{b} \in \mathbb{R}^n$ is called a **linear combination** of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ if there exist scalars $x_1, \ldots, x_n$ such that

$$\vec{b} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

Facts:

(1) If a linear system is consistent, then it has either
- Infinitely many solutions (in this case, there is at least one free variable) or
- exactly one solution (in this case, all the variables are leading (correspond to pivot rows).

(2) Suppose that A is the an $n \times m$ matrix representing a system of n linear equations in m variables. Then
- $\operatorname{rank}(A) \leqslant n$ and $\operatorname{rank}(A) \leqslant m$.
- If $\operatorname{rank}(A) = n$ then the system is consistent.
- If $\operatorname{rank}(A) = m$, then the system has at most one solution.
- If $\operatorname{rank}(A) < m$, then the system has either infinitely many solutions, or none.

(3) A linear system with fewer equations than unknowns has either no solutions or infinitely many solutions.

(4) A linear system of n equations in n variables has a unique solution iff the rank of its coefficient matrix A is n. In this case, $\operatorname{rref}(A) = I_n$.

(5) If the column vectors of an $n \times m$ matrix A are $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{x}$ is a vector in $\mathbb{R}^m$ with components $x_1, \ldots, x_m$, then

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

(6) If A is an $n \times m$ matrix, $\vec{x}, \vec{y} \in \mathbb{R}^m$, and $k \in \mathbb{R}$, then
- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$.
- $A(k\vec{x}) = k\,(A\vec{x})$.

Week 3

Definitions

(1) Consider two sets X and Y. A **function** T from X to Y is a rule that associates with each element $x \in X$ a unique element $y \in Y$. The set X is called the **domain** or **input** of the function T. The set Y is called the **target space, codomain,** or **output** of T.

(2) A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is called a **linear transformation** if there exists an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x}$ in the vector space $\mathbb{R}^m$.

(3) **Matrix Multiplication:**
   (a) Let B be an $n \times p$ matrix and A a $q \times m$ matrix. The **product** BA is defined if and only of $p = q$.
   (b) If B is an $n \times p$ matrix and A is a $p \times m$ matrix, then the product BA is defined as the matrix of the linear transformation $T(\vec{x}) = B\,(A\vec{x})$.
   (c) Suppose that A and B are two matrices for which AB and BA are both defined. If $AB = BA$ then we say that A **commutes** with B.

(4) A function $T : X \to Y$ is called **invertible** if the equation $T(x) = y$ has a unique solution $x \in X$ for each $y \in Y$. In this case, the **inverse** $T^{-1} : Y \to X$ is defined by

$$T^{-1}(y) = \text{the unique } x \in X \text{ such that } T(x) = y.$$

Facts

(1) Consider a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$. Then the matrix of T is

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix}$$

where $\vec{e}_i$ is the vector in $\mathbb{R}^m$ whose components are all zero except for the ith component which is equal to 1.

(2) A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear iff
- $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^m$, and
- $T(k\vec{v}) = kT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^m$ and $k \in \mathbb{R}$.

(3) In general, it is *not the case* that $AB = BA$. That is, matrix multiplication is **non-commutative**.

(4) For any $n \times m$ matrix A,

$$AI_m = I_nA = A.$$

(5) Matrix multiplication satisfies the following properties
  (a) $(AB)C = A(BC)$ (associativity)
  (b) $A(C + D) = AC + AD$ and $(A + B)C = AC + BC$ (distributivity)
  (c) $(kA)B = A(kB) = k(AB)$ (commutativity of scalar multiplication).

(6) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates a vector $\vec{v}$ *counterclockwise* by an angle θ. The matrix which represents this rotation is

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

To rotate in the *clockwise* direction, multiply the vector by

$$R_{-\theta} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

(7) Suppose a line L passes through the origin and makes an angle θ with the x-axis. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects a vector $\vec{v}$ over the line L. The matrix which represents this reflection is

$$B_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}.$$

Week 4

Definitions

(1) A function $T : X \to Y$ is called **invertible** if the equation $T(x) = y$ has a unique solution $x \in X$ for each $y \in Y$. In this case, the **inverse** $T^{-1} : Y \to X$ is defined by

$$T^{-1}(y) = \text{the unique } x \in X \text{ such that } T(x) = y.$$

(2) A square matrix A is called **invertible** if the linear transformation T defined by $T(\vec{x}) = A\vec{x}$ is invertible.

(3) Suppose that A is an invertible square matrix. Then the **inverse matrix of** A denoted by A^{-1} is the unique square matrix which satisfies

$$AA^{-1} = A^{-1}A = I_n.$$

For $T(\vec{x}) = A\vec{x}$, we also have $T^{-1}(\vec{x}) = A^{-1}\vec{x}$.

(4) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The quantity $ad - bc$ is called the **determinant** of the matrix A.

(5) The **image** of a function $f : X \to Y$ is

$$\text{image}(f) = \{b \in Y \mid b = f(x) \text{ for some } x \in X\}.$$

Similarly, the **image of an n by m matrix** A is

$$\text{image}(A) = \{\vec{y} \in \mathbb{R}^n \mid \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^m\}.$$

(6) Consider a set of vectors $\{v_1, \ldots, v_m\}$ in $\mathbb{R}^n$. The set of *all* linear combinations $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ is called the **span** of this set. We write

$$\operatorname{span}\{v_1, \ldots, v_m\} = \left\{ \sum_{i=1}^{m} c_i\vec{v}_i \mid c_i \in \mathbb{R}, \vec{v}_i \in \{\vec{v}_1, \ldots, \vec{v}_m\} \right\}.$$

(7) The **kernel** or **nullspace** of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\ker(T) = \left\{ \vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0} \in \mathbb{R}^n \right\}.$$

Similarly, the **kernel** of an $n \times m$ matrix A is

$$\ker(A) = \left\{ \vec{x} \in \mathbb{R}^m \mid A\vec{x} = \vec{0} \in \mathbb{R}^n \right\}.$$

Facts

(1) Let A be an $n \times n$ matrix. TFAE (The following are equivalent).
- A is invertible
- $\operatorname{rref}(A) = I_n$
- $\operatorname{rank}(A) = n$.
- $A\vec{x} = 0$ has exactly one solution (which turns out to be $\vec{x} = \vec{0}$).
- $A\vec{x} = \vec{b}$ has exactly one solution (which turns out to be $A^{-1}\vec{b}$).

(2) Let A be an $n \times n$ matrix that is *not* invertible. Then $A\vec{x} = \vec{b}$ has infinitely many solutions or none.

(3) To find the *inverse* of an $n \times n$ matrix, compute $\operatorname{rref}[A|I_n]$.
- If $\operatorname{rref}[A|I_n]$ is of the form $[I_n|B]$ then A is invertible and $A^{-1} = B$.
- If $\operatorname{rref}[A|I_n]$ is of any other form, A is not invertible.

(4) If A and B are invertible $n \times n$ matrices then BA is invertible as well and

$$(BA)^{-1} = A^{-1}B^{-1}.$$

(5) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(6) The image of a linear transformation $T(\vec{x}) = A\vec{x}$ is the span of the column vectors of A.

(7) Suppose that A is an $n \times m$ matrix. Then

$$\ker A = \left\{\vec{0}\right\} \iff \operatorname{rank}(A) = m$$

(8) Suppose that A is $n \times n$. Then

$$\ker A = \left\{\vec{0}\right\} \iff A \text{ is invertible}$$

Week 5

Definitions

(1) A set $W \subseteq \mathbb{R}^n$ is said to be **closed under addition** if

$$\vec{w}_1, \vec{w}_2 \in W \quad \Longrightarrow \quad \vec{w}_1 + \vec{w}_2 \in W.$$

(2) A set $W \subseteq \mathbb{R}^n$ is said to be **closed under scalar multiplication** if

$$\vec{w} \in W \quad \Longrightarrow \quad k\vec{w} \in W \ \forall\, k \in \mathbb{R}.$$

(3) A subset W of $\mathbb{R}^n$ is called a **linear subspace** if
   (a) $\vec{0} \in W$.
   (b) W is closed under addition
   (c) W is closed under scalar multiplication.

(4) Consider the set of vectors $\{\vec{v}_1, \ldots, \vec{v}_m\}$ in $\mathbb{R}^n$.
   (a) $\vec{v}_i$ in the list $\vec{v}_1, \ldots, \vec{v}_k$ is called **redundant** if $\vec{v}_i$ is a linear combination of the preceding vectors $\vec{v}_1, \ldots, \vec{v}_{i-1}$.

   (b) the vectors $\vec{v}_1, \ldots, \vec{v}_m$ are called **linearly independent** if none of them is redundant.
(5) The set of vectors $\{\vec{v}_1, \ldots, \vec{v}_m\}$ form a **basis** of a subspace V of $\mathbb{R}^n$ if
   - $V = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\}$ and
   - $\{\vec{v}_1, \ldots, \vec{v}_m\}$ are linearly independent.
(6) Consider the set of vectors $\{\vec{v}_1, \ldots, \vec{v}_m\}$. An equation of the form
$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$$
where $c_i \in \mathbb{R}$ is called a **relation** among the vectors $\vec{v}_1, \ldots, \vec{v}_m$. A relation is called **trivial** if $c_1 = \cdots = c_m = 0$ and **non-trivial** if it is not trivial.
(7) The number of elements in a basis of a linear subspace $V \subseteq \mathbb{R}^n$ is called the **dimension** of V.
(8) (Optional) Suppose that W is an **affine space**. That is, each element $\vec{w} \in W$ is of the form $\vec{w} = \vec{x}_0 + \vec{v}$ for some fixed $\vec{x}_0$ and some $\vec{v}$ in a vector space V. Then the **dimension** of W is just the dimension of V.

Facts

(1) The image of a linear transformation $T(\vec{x}) = A\vec{x}$ is the span of the column vectors of A.
(2) The image of a linear transformation $T : R^m \to \mathbb{R}^n$ has the following properties
   (a) $\vec{0} \in \operatorname{image} T$
   (b) $\operatorname{image} T$ is *closed under addition.*
   (c) $\operatorname{image} T$ is *closed under scalar multiplication.*
   Thus $\operatorname{image} T$ is a *subspace* of $\mathbb{R}^n$.
(3) Consider a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$.
   (a) $\vec{0} \in \ker T$.
   (b) $\ker T$ is closed under addition
   (c) $\ker T$ is closed under scalar multiplication.
   Thus $\ker T$ is a subspace of $\mathbb{R}^m$.
(4) All bases of a subspace $V \subseteq \mathbb{R}^n$ have the same number of elements.
(5) Consider an m-dimensional subspace V of $\mathbb{R}^n$.
   (a) One can find *at most* m linearly independent vectors in V.
   (b) We need *at least* m vectors to span V.
   (c) If m vectors in V are linearly independent, then they form a basis of V.
   (d) If m vectors in V span V then they form a basis for V.
(6) For any matrix A,
$$\dim \operatorname{image}(A) = \operatorname{rank}(A)$$
(7) **Rank–Nullity Theorem:** For any $n \times m$ matrix A,
$$\dim \ker A + \dim \operatorname{image} A = m.$$
In other words,
$$\text{nullity of } A + \operatorname{rank} A = m.$$
(8) The vectors $\{\vec{v}_1, \ldots, \vec{v}_m\}$ form a basis of $\mathbb{R}^n$ iff the matrix
$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix}$$
is invertible.

Week 6

Definitions

(1) Consider a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ of a subspace $V \subseteq \mathbb{R}^n$. Then any vector $\vec{x} \in V$ can be written uniquely as
$$\vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m.$$

The scalars $c_1, c_2, \dots, c_m$ are called the $\mathcal{B}$**-coordinates** of $\vec{x}$ and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

is called the $\mathcal{B}$**-coordinate vector of** $\vec{x}$ and is denoted by $[\vec{x}]_{\mathcal{B}}$.

(2) Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathcal{B}$ of $\mathbb{R}^n$. The $n \times n$ matrix B satisfying

$$[T(\vec{x})]_{\mathcal{B}} = B\,[\vec{x}]_{\mathcal{B}}$$

for all $\vec{x} \in \mathbb{R}^n$ is called the $\mathcal{B}$**-matrix** of T.

(3) Let V be an m-dimensional subspace of $\mathbb{R}^n$, and let $\mathcal{B} = \{b_1, \dots, b_m\}$ and $\mathcal{C} = \{c_1, \dots, c_n\}$ be bases of V. The *change of basis matrix from* $\mathcal{C}$ *to* $\mathcal{B}$, denoted $S_{\mathcal{B}\leftarrow\mathcal{C}}$, is the unique $n \times n$ matrix such that

$$[\vec{v}]_{\mathcal{B}} \;=\; S_{\mathcal{B}\leftarrow\mathcal{C}}\,[\vec{v}]_{\mathcal{C}} \quad \text{for all } \vec{v} \in V.$$

(4) Two matrices A, B are said to be **similar** if there exists an invertible matrix M such that

$$A = MBM^{-1}.$$

(5) Suppose that $W \subseteq \mathbb{R}^n$ is a subspace. It's *orthogonal complement* is the set $W^\perp$ defined by

$$W^\perp = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \;\forall\, \vec{w} \in W\}$$

Facts

(1) Let $W \subseteq \mathbb{R}^n$ be a subspace. Then $\dim W^\perp = n - \dim W$.

(2) Let A be an $m \times n$ matrix. Then $(\ker A)^\perp = \operatorname{Im}\left(A^\top\right)$.

(3) Let V be a linear subspace with two given bases $\mathcal{B}$ and $\mathcal{B}'$. Consider a linear transformation $T : V \to V$ and let A, B be the $\mathcal{B}$- and $\mathcal{B}'$-matrix of T, respectively. Let S be the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$. Then A is similar to B and

$$AS = SB \iff A = SBS^{-1} \iff B = S^{-1}AS.$$

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